The Use of Fractal Dimension in Arts Analysis

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Abstract

The fractal dimension is a measure for the coarseness of objects and textures. This property can be assigned to graylevel images to get a measure for the coarseness of a texture in an image, [2]. The theme of this article is the application of fractal dimension to arts analysis. Considering 7 artists, everyone represented by about 18 images, we searched for qualified description parameters for the painters characteristics. We evaluated among other feature values 3 different types of fractal dimension, namely capacity (or box counting) dimension, information dimension and correlation dimension, [3]. With a special feature selection algorithm the best features for the classification of the images with respect to the painter were evaluated, [5]. The fractal dimension turned out to be under the best features. In the following the computation of the fractal dimension, our classification task and a few results are discussed.

1. Fractal Dimension

We want to define the notion of a dimension of an image. From the mathematical point of view the best way would be to take the Hausdorff dimension [1], since this exists for all (bounded) subsets of a metric space, and an image is a bounded subset of Euclidean 3–dimensional space (at least a graylevel image). But the Hausdorff dimension has a complicated definition and it is by no means easy to calculate it. Therefore simpler versions for the notion of a dimension were suggested. A very useful notion goes back to A. Renyi [4], the generalized q–dimension. We will start with one version of this q{dimension and will specialise then to its calculation for images.

1.1. The generalized *q*–dimension of A. Renyi

Let A be a bounded, measurable subset of the 3-dimensional space $\Re^3$. For $\varepsilon > 0$ we consider a lattice of cubes of side length ε in $\Re^3$ call these cubes C_1; C_2; .... Then let

$$p_i(\varepsilon) := \frac{measure(A \cap C_i)}{measure(A)},$$

(this gives a probability measure on A. To be more precise $p_i(\varepsilon) :=$ Prob (a point of A lies in C_i)).

Definition 1.1 *For* $q \geq 0$ *the generalized q–dimension of A is defined as*

$$D_q(A) := \lim_{\varepsilon \to 0} \frac{1}{q-1} \cdot \frac{\ln \sum_i p_i(\varepsilon)^q}{\ln \varepsilon},$$

(if the limit exists).

Remark 1.2 *It can be shown that $D_q(A)$ is well defined, i.e. independent of the choice of the cubic lattice (origin and direction of axes) and that $D_q(A)$ is a decreasing function with respect to q. Furthermore for* $A \subset \Re^3$ *one has* $0 \leq D_q(A) \leq 3$.

Important special cases are:

***q* = 0**: Here we use the convention that $0^0 = 0$. Then $\sum (p_i(\varepsilon))^0 =$ (number of cubes ofside length ε that contain a part of positive measure from $A) = N(A, \varepsilon)$.

So we get from the definition above

$$D_0(A) = \lim_{\varepsilon \to 0} \frac{1}{q-1} \cdot \frac{\ln \sum_i N(A,\varepsilon)}{\ln 1/\varepsilon} =: D_C(A),$$

which is usually called the *fractal dimension* of *A* (or also capacity dimension of *A*).

***q* = 2**: The definition gives

$$D_2(A) := \lim_{\varepsilon \to 0} \frac{\ln \sum_i p_i(\varepsilon)^2}{\ln \varepsilon} =: D_{corr}(A),$$

the so called *correlation dimension* of *A*. The expression in the nominator has a nice interpretation, namely $\sum (p_i(\varepsilon))^2$ =Prob (2 points of *A* lie in the same cube C_i).

***q* = 1**: In this case we have to be careful since for $q = 1$ the nominator and the denominator in the definition of $D_q(A)$ vanish (apply de l'Hospital).

$$D_1(A) := \lim_{\varepsilon \to 0} \frac{1}{\ln \varepsilon} \lim_{q \to 1} \frac{\ln \sum_i p_i(\varepsilon)^q}{q-1} = \lim_{\varepsilon \to 0} \frac{\ln \sum_i p_i(\varepsilon) \ln p_i(\varepsilon)}{\ln \varepsilon},$$

which is generally called the information dimension of *A* (since the nominator corresponds to the entropy, which is a measure for information).

1.2. Practical calculation of the dimension of an image

An image is a finite set of points (= pixels with certain grey values). Unfortunately one can easily show:

$$D_q(\text{finite set}) = 0,$$

So the above definitions cannot be applied directly, since the value 0 for a dimension is not very interesting. Taking another interpretation of a pixel as a square of side length 1 (say) then the limit $\varepsilon \to 0$ makes no sense (ε should be ≥ 1 in this case). Therefore we write the definition for $D_q(A)$ in the following way:

$$D_q = D_q(A) = \frac{1}{q-1} \frac{\ln \sum_i p_i(\varepsilon)^q}{\ln \varepsilon} + \frac{\ln(1/r)}{(q-1)\ln \varepsilon},$$

(where $r = r(A,\varepsilon)$ is an error term). Multiplying with the denominator gives

$$D_q(q-1)\ln \varepsilon = \ln \sum_i p_i(\varepsilon)^q + \ln(1/r),$$

or if we write $x_c = (q-1)\ln \varepsilon$; $y_c = \ln \sum_i p_i(\varepsilon)^q$, we get

$$y_c = D_q x_c + \ln r.$$

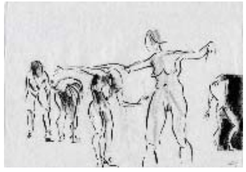
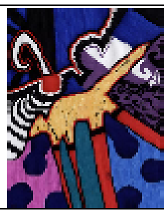
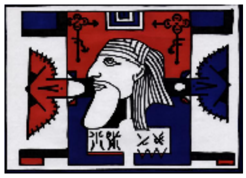
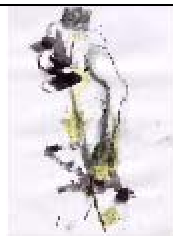
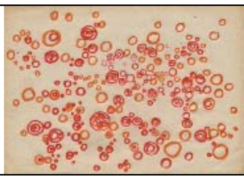
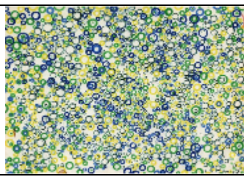
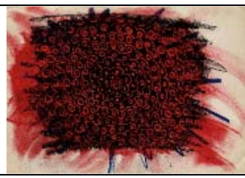
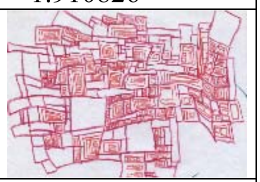
This is the equation of a straight line (in a logarithmic coordinate system) with respect to a fixed lattice of squares or cubes and we are interested into its slope D_q. Practically we choose several values for ε, calculate (from the image *A*) the probabilities

$$p_i(\varepsilon) = \frac{\#(\text{pixels of } A \text{ in } C_i)}{\#(\text{pixels of } A)},$$

and get for each $\varepsilon > 0$ a point $(x_\varepsilon, y_\varepsilon)$. The slope of the regression line defined by these points gives finally the *q*–dimension $D_q(A)$.

2. Results

In the following table 20 sample images of 5 different painters are shown and the respective fractal dimensions are listed. Values of the capacity dimension and information dimension of the graylevel version of the images (capdim gray, infdim gray) and the capacity dimension of the binary version (capdim bin) are presented. To obtain the anonymity of the artists, we used acronyms instead of their names.

painter bg				
capdim gray	2.246206	2.091211	2.270820	2.203044
infdim gray	2.281946	2.082100	2.278741	2.241939
capdim bin	1.525424	1.574814	1.674945	1.430831
painter mk				
capdim gray	2.227004	2.237732	2.250339	2.225809
infdim gray	2.182206	2.193382	2.214684	2.165344
capdim bin	1.990236	2.050675	2.012956	1.976064
painter if				
capdim gray	2.243127	2.234009	2.229157	2.221351
infdim gray	2.223151	2.205055	2.189209	2.198305
capdim bin	1.814990	1.797485	1.943745	1.705103
painter ve				
capdim gray	2.229848	2.284704	2.237109	2.283854
infdim gray	2.181884	2.264767	2.221189	2.266364
capdim bin	1.915681	1.850409	1.797892	1.910826
painter wi				
capdim gray	2.299416	2.368323	2.327271	2.335588
infdim gray	2.272672	2.346169	2.290355	2.355060
capdim bin	1.799837	1.969536	1.983552	1.901535

3. Feature Selection and Image Classification

In our method we use first and second order statistical data to build a feature-space representation for various painters. We combined up to five features and obtained for each feature combination a feature-space which is divided into separate classes representing our painters. In order to extract the best feature combination, we computed the distances of the feature values of each image of a painter to the mean values, the centre of each class. We chose the Mahalanobis distance as our probabilistic distance measure. Afterwards, we noticed the class to which each feature vector (representing an image) of a painter has minimal distance. Of course, the classification is correct if the feature vector, the image, has minimal distance to its painter. Thus we obtain the best feature combination for the maximum classification rate.

With these best features we try to identify the painter of an unseen image. To improve the classification results we cluster pictures of the painters before classification with respect to different styles, see [5].

For illustration we consider 7 painters. Every painter is represented by 16 to 20 images. As an example we tried to classify these images in a 4{dimensional feature space. In the following table a few results for the percentage of the correct classified pictures from a set of 128 pictures are shown. Values of the capacity dimension of the graylevel version of the images (capdim gray) and the capacity dimension of the binary version (capdim bin) are used as well as statistics, like entropy, mean and variance, of the graylevel distribution in the hue image (hue entropy, hue mean, hue variance).

feature1	feature2	feature3	feature4	percentage of correct classified pictures
capdim bin	capdim gray	hue entropy	hue variance	75 percent
capdim bin	capdim gray	hue mean	hue variance	80 percent
capdim bin	capdim gray	hue entropy	hue mean	70 percent

4. References

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