Page 70, equation (3.17)

It is sufficient to show the derivation for the case $\det \mathbf{F} > 0$.

Cf. also Footnote 4 on page 70 in the book.

Eq. (3.5) tells us that $\dot{J}$ is the following partial derivative:

$$\dot{J} = \frac{\partial J(\mathbf{X}, t)}{\partial t}$$

which can be expressed:

$$\frac{\partial J(\mathbf{X}, t)}{\partial t} = \frac{\partial J}{\partial \mathbf{F}} \cdot \frac{\partial \mathbf{F}(\mathbf{X}, t)}{\partial t} = \frac{\partial J}{\partial F^{iJ}} \frac{\partial F^{iJ}}{\partial t} = \operatorname{tr}\left[\frac{\partial J}{\partial \mathbf{F}} \left(\frac{\partial \mathbf{F}}{\partial t}\right)^T\right] \tag{1}$$

(Einstein summation convention is used in the third expression). The expression $\partial J/\partial F^{iJ}$ is the derivative of determinant with respect to one of its elements: $\partial \det \mathbf{F}/\partial F^{iJ}$. Remind that tensor $\mathbf{A}$ can be expanded in minors M_{ij}:

$$\det \mathbf{A} = \sum_j (-1)^{i+j} A^{ij} M_{ij}$$

for arbitrary i, j. Then

$$\frac{\partial \det \mathbf{A}}{\partial A^{ij}} = (-1)^{i+j} M_{ij}. \tag{2}$$

Remember also that inverse of $\mathbf{A}$ is related to determinant as follows:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

where the cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ and so $A^{-1^{ij}} = C_{ji}/\det \mathbf{A}$ and, finally, it follows from (2):

$$\frac{\partial \det \mathbf{A}}{\partial A^{ij}} = \det \mathbf{A}\, A^{-1^{ji}} \Rightarrow \frac{\partial \det \mathbf{A}}{\partial \mathbf{A}} = (\det \mathbf{A})(\mathbf{A}^{-1})^T. \tag{3}$$

Capitalizing on (1) and (3) we can write for the partial derivative:

$$\dot{J} = \operatorname{tr}\left[\det \mathbf{F} (\mathbf{F}^{-1})^T \left(\frac{\partial \mathbf{F}}{\partial t}\right)^T\right] = \det \mathbf{F} \operatorname{tr}\left(\frac{\partial \mathbf{F}}{\partial t} \mathbf{F}^{-1}\right). \tag{4}$$

Successive substitution of (3.12), (3.14) and (3.16) into (4) gives:

$$\dot{J} = J \operatorname{tr}(\dot{\mathbf{F}} \mathbf{F}^{-1}) = J \operatorname{tr} \mathbf{L} = J \operatorname{div} \mathbf{v}$$

and this is (3.17).