Exercise 2 to section 3.7.[1] Heat conduction II

Derive the (differential) equation for the temperature change in an isobaric fluid with no (internal) friction and radiation. Use the results of sec. 3.7.

Try to answer before continuing reading.

The stress tensor (3.195) is reduced to $\mathbf{T} = -P\mathbf{1}$. The energy balance (3.107) then looks:

$$\rho\dot{u} = -\mathrm{div}\mathbf{q} + \mathrm{tr}(\mathbf{TD}) = -\mathrm{div}\mathbf{q} - P\,\mathrm{tr}\mathbf{D}. \tag{1}$$

From the mass balance (3.63) it follows that $\mathrm{div}\mathbf{v} = -\dot{\rho}/\rho$ and, upon combining with (3.16), $\mathrm{tr}\mathbf{D} = -\dot{\rho}/\rho$. Definition (3.199) gives $\dot{v} = -(1/\rho^2)\dot{\rho}$, so $\mathrm{tr}\mathbf{D} = \rho\dot{v}$ and $-P\mathrm{tr}\mathbf{D} = -P\rho\dot{v}$.[2] The balance (1) can be thus rewritten:

$$\rho\dot{u} = -\mathrm{div}\mathbf{q} - P\rho\dot{v}. \tag{2}$$

Substituting from Fourier law (3.187) into (2) we have

$$\rho\dot{u} = \mathrm{div}(k\mathbf{g}) - P\rho\dot{v}$$

or

$$\rho(\dot{u} + P\dot{v}) = \mathrm{div}(k\mathbf{g}). \tag{3}$$

Let us define the specific enthalpy as $h = u + Pv$ and rewrite (3):

$$\rho\dot{h} = \mathrm{div}(k\mathbf{g}). \tag{4}$$

It follows from the definition that $h = \hat{h}(\rho, T)$; by inversion of (3.194), which is justified by (3.257), this function can be transformed into $h = \tilde{h}(P, T)$. At the isobaric conditions we can write

$$\dot{h} = \frac{\partial\tilde{h}}{\partial T}\dot{T} = c_P\dot{T} \tag{5}$$

where $c_P = \partial\tilde{h}/\partial T$ is the heat capacity at constant pressure. Combining (4) and (5) we get

$$\rho c_P\dot{T} = \mathrm{div}(k\mathbf{g}). \tag{6}$$

[1] Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).

[2] Alternatively, we could use the procedure shown in the note on the volume work on this website.

Defining the temperature conductivity

$$\alpha' = \frac{k}{\rho c_P}$$

the final equation follows from (6):

$$\dot{T} = \alpha' \mathrm{div}(\mathrm{grad} T). \tag{7}$$

In the case of negligible velocity ($\mathbf{v} = \mathbf{o}$), eq. (7) is reduced to

$$\frac{\partial T}{\partial t} = \alpha' \mathrm{div}(\mathrm{grad} T). \tag{8}$$

One-dimensional version of (8) (the heat conduction equation along the coordinate x) is:

$$\frac{\partial T}{\partial t} = \alpha' \frac{\partial^2 T}{\partial x^2}.$$