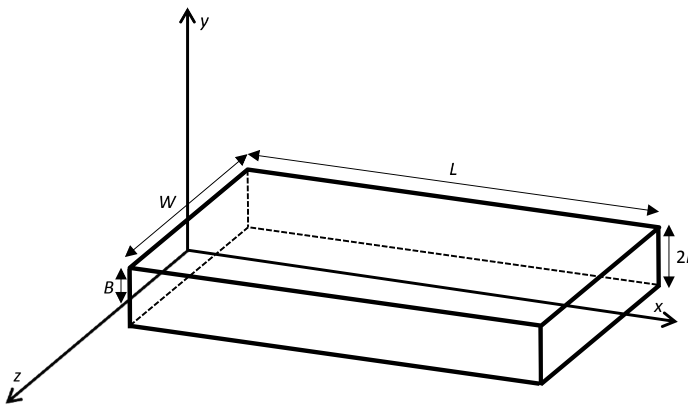
Exercise 6 to section 3.7.[1]

Incompressible fluid flows in a rectangular slit without the action of any external or inertial fields. The flow is laminar, stationary, in the direction of x-axis; the fluid clings to the slit walls. The pressure at the input is P_0, at the output P_L. See also the figure below. Integrating Navier-Stokes equation derive equation for the velocity field of the fluid, then the expressions for the maximum and mean velocity, and for the volume flow rate.

Consider a slit of the following dimensions: thickness $B = 0.5$ mm, length $L = 50$ cm, width $W = 20$ cm. For the pressure difference between the slit input and output equal to 1 kPa the volume flow rate of water (at room temperature) was determined as 0.333 cm^3/s. Calculate the viscosity of water and its maximum and average velocities.

Try to answer before continuing reading.



Navier-Stokes equation for the incompressible fluid is (cf. exercise 3 to section 3.7):

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}P + \eta \,\mathrm{div}\,\mathrm{grad}\mathbf{v} + \rho(\mathbf{b} + \mathbf{i})$$

and in our case is simplified to

$$\mathbf{o} = -\mathrm{grad}P + \eta \,\mathrm{div}\,\mathrm{grad}\mathbf{v} \tag{1}$$

(the stationary case means that $\dot{\mathbf{v}} = \mathbf{o}$).

[1]Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).

The fluid flows along the x-axis only, thus $v^y = v^z = 0$ and $v^x \neq 0$. From the incompressibility condition then follows that v^x does not depend on the x coordinate:

$$\mathrm{div}\mathbf{v} = \frac{\partial v^x}{\partial x} = 0. \tag{2}$$

The flow is planar (laminar), i.e. in the form of parallel planes, identical in all planes perpendicular to the z-axis. In other words, v^x does not depend on the z coordinate, as well:

$$v^x = v^x(y). \tag{3}$$

From Navier-Stokes equation (1) then follows:

$$\frac{\partial P}{\partial x} = \eta \frac{\partial^2 v^x}{\partial y^2} \tag{4}$$

and:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial P}{\partial z}. \tag{5}$$

Eq. (5) states that $P = P(x)$. Taking into account (3), we see that the left and right hand sides of (4) contain functions of different independent variables and thus must represent an identity of constants, for instance:

$$(1/\eta)\frac{\partial P}{\partial x} = C \,(= \text{const.}). \tag{6}$$

Eq. (6) can be integrated taking into account the following boundary conditions:

$$x = 0, P = P_0,$$

$$x = L, P = P_L.$$

The results is $P = C\eta x +$ const. where

$$C = \frac{P_L - P_0}{\eta L}. \tag{7}$$

Substitution of (6) into (4) gives:

$$\frac{\partial^2 v^x}{\partial y^2} = C. \tag{8}$$

The boundary conditions follow from the information on the sticking to the walls:

$$y = \pm B, v^x = 0. \tag{9}$$

First integration of (8) gives:

$$\frac{\partial v^x}{\partial y} = Cy + C_1. \tag{10}$$

Integrating for the second time we obtain:

$$v^x = (1/2)Cy^2 + C_1 y + C_2. \tag{11}$$

Introducing boundary conditions (9) into (11) we find that $C_1 = 0$ and $C_2 = -(1/2)CB^2$. Thus, the final result is (taking into account (7)):

$$v^x = \frac{1}{2}\frac{P_L - P_0}{\eta L}(y^2 - B^2) = \frac{(P_0 - P_L)B^2}{2\eta L}\Big(1 - \frac{y^2}{B^2}\Big). \tag{12}$$

The maximum velocity is obviously attained when $y = 0$, i.e. in the center of the slit:

$$v^x_{\text{max}} = \frac{1}{2}\frac{P_0 - P_L}{\eta L}B^2. \tag{13}$$

The average (mean) velocity is defined by integral:

$$\bar{v}^x = \frac{1}{2B}\int_{-B}^{B} v^x dy = \frac{P_0 - P_L}{3\eta L}B^2 \equiv \frac{2}{3}v^x_{\text{max}}. \tag{14}$$

The volume flow rate is then given by:

$$Q = 2BW\bar{v}^x = 2W\frac{P_0 - P_L}{3\eta L}B^2. \tag{15}$$

The viscosity of water can be calculated from the given data and Eq. (15); the result is $0.1\,\text{kg}\,\text{m}^{-1}\text{s}^{-1}$ (= 1 cP). The maximum velocity calculated from Eq. (13) is $0.0025\,\text{m}\,\text{s}^{-1}$ and the average velocity calculated from (14) is $0.00167\,\text{m}\,\text{s}^{-1}$.