Exercise 3 to section 3.1[1]

The *rigid motion* can be expressed generally in the form $\mathbf{x} = \mathbf{\Gamma}(t)\mathbf{X} + \boldsymbol{\gamma}(t)$ where $\mathbf{\Gamma}(t)$ is the orthogonal tensor function of time and $\boldsymbol{\gamma}(t)$ is the vector function of time, i.e., their values are orthogonal tensors $\mathbf{\Gamma}$ and vectors $\boldsymbol{\gamma}(t)$, respectively. Show that the deformation gradient $\mathbf{F}$ is orthogonal tensor, the velocity gradient is identical with the spin and thus Killing's theorem is fulfilled (p. 71). Try to answer before continuing reading.

The deformation gradient is in this case given by:

$$\mathbf{F} = \frac{\partial \underline{\chi}}{\partial \mathbf{X}} = \mathbf{\Gamma}. \tag{1}$$

Thus, the deformation gradient is equal to orthogonal tensor.

The velocity gradient is given by eq. (3.14) (p. 70) which is in this example:

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1} = \dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{T} \tag{2}$$

where the definition of the orthogonal tensor, $\mathbf{\Gamma}^T = \mathbf{\Gamma}^{-1}$, was used in the last equality.

The rate of deformation tensor is expressed (cf. p. 70):

$$\mathbf{D} = \tfrac{1}{2}[\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^T + (\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^T)^T] = \tfrac{1}{2}(\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^T + \mathbf{\Gamma}\dot{\mathbf{\Gamma}}^T). \tag{3}$$

Taking the time derivative of the (definition) property of an orthogonal tensor, $\mathbf{\Gamma}\mathbf{\Gamma}^T = \mathbf{\Gamma}^T\mathbf{\Gamma} = \mathbf{1}$, we have $\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^T + \mathbf{\Gamma}\dot{\mathbf{\Gamma}}^T = \mathbf{0}$. Introducing the last equality into (3) it follows that $\mathbf{D} = \mathbf{0}$ which confirms Killing's theorem and together with (3.15) shows that $\mathbf{L} = \mathbf{W}$. This completes the exercise.

[1]Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).